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INTRODUCTION

Free-flight data of pitching and yawing motions of symmetric blunt-nosed missiles at hypersonic speeds have indicated a strong dependence of the frequency of oscillation on the maximum and minimum angles of the motion (e.g., refs. 1-4). This dependence indicates the highly nonlinear nature of the pitching-moment characteristics of these missiles. A knowledge of the effects of the pitching-moment nonlinearities on the frequency of the motion is thus very important in the analysis of free-flight data and prediction of flight characteristics.

The general motion of a pitching and yawing missile attendant with all of its nonlinearities is very complicated, and a variety of assumptions and restrictions have been made by the authors who have analyzed these nonlinear effects. The most general analysis has been made by Murphy, who has examined nonlinearities associated with damped motions as well as stationary motions. In reference 1, Murphy has considered the motion of a missile with a restoring moment described by a polynomial of odd powers of the angle of attack. The nonlinearities were assumed to be small, and Kryloff-Bogoliuboff techniques were applied to the equations of motion. In reference 2, Murphy applied a perturbation technique for a polynomial moment of the first and third powers of the angle of attack and removed the restriction of quasi-linear motion. In reference 3, Rasmussen obtained exact solutions for a first and third power cubic moment for zero damping. Kirk, in reference 4, developed exact solutions for a one-term moment of arbitrary power of angle of attack, a cubic moment with first, second, and third powers of angle of attack, and a quintic moment of first, third, and fifth powers of angle of attack. Kirk, however, considered only planar motion with zero damping.

The pitching-moment characteristics of symmetric missiles are deduced by fitting the free-flight data with specific solutions to the equations of motion. Rather than the data itself suggesting the form of the pitching moment, the data reduction is dependent upon the particular assumptions adopted for the analysis of the equations of motion. An important step, therefore, toward a more general reduction of free-flight data would be an analysis of the pitching and yawing motion due to an arbitrary restoring moment.

The object of this report is to present an analysis of the conservative pitching and yawing motion of a spinning symmetric missile acted upon by an arbitrary restoring moment. The restoring moment will be represented by an arbitrary power series of the resultant angle of attack. In addition to the nonlinear terms included in the restoring moment, the nonlinear inertial terms due to spin will also be included in the analysis.

NOMENCLATURE

A	reference area
a_n	constants defined by equation (14)
b	constant of the motion
C_m	restoring-moment coefficient, $\frac{\text{restoring moment}}{qAl}$
$I_1, I_{y'}, I_{z'}$	moments of inertia about pitch and yaw axes
$I_3, I_{x'}$	moment of inertia about roll axis
L	Lagrangian, T-V
l	reference length
M	$\frac{qAl}{I_1} C_m$
M_n	restoring-moment coefficients, defined by equation (9)
$\overline{M}_n$	$M_n + \frac{n + 2}{2} a_{n+4}$
P	$\frac{I_3}{I_1} \omega_{x'}$ (gyroscopic spin)
q	dynamic pressure, $\frac{1}{2} \rho v^2$
T	kinetic energy of the motion
T_o	period of oscillation
t	time
V	potential energy of the motion
v	linear velocity of the missile
α	angle of attack
α_m	maximum angle of attack, planar motion
β	angle of yaw
Γ	gamma function
ϵ	resultant angle of attack squared, σ^2
$\bar{\epsilon}$	$\frac{\sigma_m^2 + \sigma_o^2}{2}$

ϵ_1	$\frac{\sigma_m^2 - \sigma_o^2}{2}$
ρ	air density
θ,σ,ψ	Euler angles
σ	resultant angle of attack, $\tan^{-1}\sqrt{\tan^2 \alpha + \tan^2 \beta}$
σ_m	maximum resultant angle of attack
σ_o	minimum resultant angle of attack
φ	$\epsilon - \bar{\epsilon}$
ω	frequency, radians per unit time
$\omega_{x'}$	spinning velocity about roll axis
$(\dot{\ })$	$\frac{d}{dt}(\ )$
$(\ddot{\ })$	$\frac{d^2}{dt^2}(\ )$

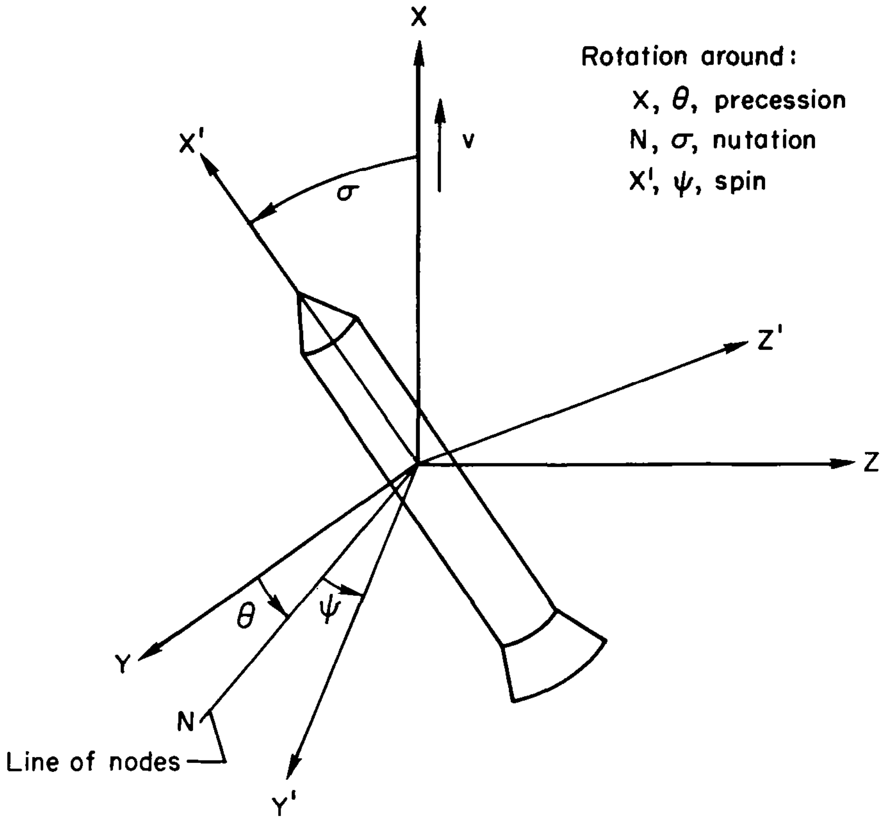
ANALYSIS

Equations of Motion

The general equations of motion for a spinning symmetric missile may be derived from different points of view. In reference 1, the equations of motion were derived for small angles of attack but included nonlinearities in the pitching moment. In this analysis, it is desired to consider the effects of large angles of attack. These effects come from the nonlinearities in the pitching moment and also from the inertial effects due to spin. Since the missile under consideration is symmetric, Lagrangian methods are appropriate for the derivation of the equations of motion. The coordinates of the motion will be described by Euler angles, and the derivation will be similar to the classical derivation of the equations of motion for a heavy top, such as given in reference 5.

Let us consider the motion of a spinning symmetric missile with its center of gravity traveling in the x direction with a constant velocity v. We will assume that the effects on the frequency of the motion of the resultant forces acting on the missile are negligible, and thus that only a pure moment about the center of gravity need be considered. Furthermore, let us assume that the moment can be derived from a potential. Thus, the motion is conservative. Finally, we will assume that the moment depends only upon the resultant angle of attack σ.

The body-fixed coordinate system will be denoted by x', y', z', with the origin at the center of gravity. This system is related to the inertial coordinate system (x, y, z) by the Euler angles θ, σ, and ψ (sketch (a)).



Sketch (a)

The kinetic energy of motion is given in reference 5 in terms of the Euler angles by the expression

$$T = \frac{I_1}{2}\left(\dot{\sigma}^2 + \dot{\theta}^2 \sin^2 \sigma\right) + \frac{I_3}{2}\left(\dot{\psi} + \dot{\theta} \cos \sigma\right)^2 \tag{1}$$

where for a symmetric missile the moments of inertia are given by $I_{y'} = I_{z'} = I_1$ and $I_{x'} = I_3$.

If we denote the potential energy by $V(\sigma)$, then the Lagrangian is given by

$$L = T - V(\sigma) \tag{2}$$

Accordingly, the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{3}$$

Here the generalized coordinates q_i are the Euler angles θ, σ, and ψ.

From equations (1) and (2), we can see that the Lagrangian is not a function of θ or ψ. These coordinates are thus cyclic, and two first integrals of the motion can be written:

Angular momentum about x' axis = constant:

$$\frac{\partial L}{\partial \dot{\psi}} = I_3(\dot{\psi} + \dot{\theta} \cos \sigma) = I_3 \omega_{x'} = I_1 P \tag{4}$$

Angular momentum about x axis = constant:

$$\frac{\partial L}{\partial \dot{\theta}} = (I_1 \sin^2 \sigma + I_3 \cos^2 \sigma)\dot{\theta} + I_3 \dot{\psi} \cos \sigma = I_1 b \tag{5}$$

where P and b are two constants of the motion. The constant $P = (I_3/I_1)\omega_{x'}$ represents the spinning rate of the missile.

There is also another first integral of the motion; since the motion is conservative, the total energy $E = T + V(\sigma)$ is a constant.

$$E = \frac{I_1}{2}(\dot{\sigma}^2 + \dot{\theta}^2 \sin^2 \sigma) + \frac{I_3}{2}\omega_{x'}{}^2 + V(\sigma)$$

Since $\omega_{x'}$ is a constant, the above expression can be written

$$E_o = \frac{I_1}{2}(\dot{\sigma}^2 + \dot{\theta}^2 \sin^2 \sigma) + V(\sigma) \tag{6}$$

where $E_o = E - (1/2)I_3 \omega_{x'}{}^2$.

The third equation of motion may be determined from Lagrange's equation (3) for $q_i = \sigma$. This equation becomes

$$I_1 \ddot{\sigma} - I_1 \dot{\theta}^2 \sin \sigma \cos \sigma + I_3(\dot{\psi} + \dot{\theta} \cos \sigma)\dot{\theta} \sin \sigma + \frac{dV}{d\sigma} = 0$$

Using equation (4), we can reduce the above equation to

$$\ddot{\sigma} - \dot{\theta}^2 \sin \sigma \cos \sigma + P\dot{\theta} \sin \sigma + \frac{1}{I_1}\frac{dV}{d\sigma} = 0 \tag{7}$$

From equation (7), we can identify $dV/d\sigma$ with the restoring moment acting upon the missile:

$$\frac{1}{I_1}\frac{dV}{d\sigma} = -M(\sigma) = -\frac{qAl}{I_1} C_m \tag{8}$$

For an arbitrary moment we can write[1]

$$M(\sigma) = -\sum_{n=o} M_n \sigma^{n+1} \tag{9}$$

where Σ represents an arbitrary summation of the terms $M_n\sigma^{n+1}$. Integrating equation (9), we obtain the expression for the potential $V(\sigma)$

$$\frac{1}{I_1} V(\sigma) = \sum_{n=o} \frac{M_n}{n+2} \sigma^{n+2} \tag{10}$$

The moment, and thus the potential, may be chosen arbitrarily by an arbitrary selection of the parameters M_n.

The value of $\dot{\theta}$ may be determined from equations (4) and (5) by eliminating $\dot{\psi}$

$$\dot{\theta} = \frac{b - P\cos\sigma}{\sin^2\sigma} \tag{11}$$

Equation (11) may be substituted into equation (6) to obtain the differential equation for the resultant angle of attack σ

$$\dot{\sigma}^2 + \frac{(b - P\cos\sigma)^2}{\sin^2\sigma} + \frac{2}{I_1}V(\sigma) - E' = 0 \tag{12}$$

where $E' = (2/I_1)E_o$. Multiplying equation (12) by σ^2, we can rewrite it in the following form

$$\frac{1}{4}\left(\frac{d\sigma^2}{dt}\right)^2 = -\frac{\sigma^2}{\sin^2\sigma}(b - P\cos\sigma)^2 - \frac{2}{I_1}V(\sigma)\sigma^2 + E'\sigma^2 \tag{13}$$

Equation (13) represents the basic equation of this report. From this equation, an expression for the frequency of oscillating motion can be obtained. Equation (13) may be expressed in a more convenient form for our purposes by writing the first term on the right-hand side in terms of a power series

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[1]It should be kept in mind that for the symmetric missile under consideration, the desired moment is an odd function of angle of attack. This poses no problems in the analysis that follows (σ as defined is always ≥ 0), but limits applications to this class of bodies.

$$\frac{\sigma^2}{\sin^2 \sigma} (b - P \cos \sigma)^2 = \sum_{n=o}^{\infty} a_n \sigma^n \tag{14}$$

where $a_1 = a_3 = a_5 = a_{odd} = 0$. The values of a_n for the first several terms are written below:

$$a_o = (b - P)^2$$

$$a_2 = \frac{b^2 + P^2}{3} + \frac{bP}{3} - P^2$$

$$a_4 = \frac{b^2 + P^2}{15} + \frac{7}{60} bP$$

$$a_6 = \frac{2(b^2 + P^2)}{189} + \frac{31}{1512} bP$$

$$a_8 = \frac{b^2 + P^2}{675} + \frac{127}{43200} bP$$

Equation (13) may now be written as

$$\frac{1}{4} \left(\frac{d\sigma^2}{dt}\right)^2 = - \sum_{n=o}^{\infty} a_n \sigma^n - 2 \sum_{n=o} \frac{M_n}{n+2} \sigma^{n+4} + \sigma^2 E' \tag{15}$$

By changing the summing notation slightly, we can write the above expression as

$$\frac{1}{4} \left(\frac{d\sigma^2}{dt}\right)^2 = -a_o + (E' - a_2)\sigma^2 - 2 \sum_{n=o}^{\infty} \frac{\bar{M}_n}{n+2} \sigma^{n+4} \tag{16}$$

where

$$\bar{M}_n = M_n + \frac{n + 2}{2} a_{n+4}$$

The right side of equation (16) is expressed as an infinite series, which may be evaluated approximately to any order. This form for equation (16) shows the relation between the nonlinear terms due to the restoring moment and those due to spin. The nonlinear terms due to spin show their effects through the parameters b and P in the coefficients a_n. For large spinning rates P, these effects become important, and they vanish for planar motion. These effects may also become important at large angles of attack.

Equations (13) and (16) are expressed in terms of the three constants of the motion b, P, and E'. While these constants represent the conservation of angular momentum and energy, they do not directly express physical observables of the motion. Two of these constants, however, can be replaced by two other constants which are physically observable. These two constants are the maximum and minimum resultant angles of attack, σ_m and σ_o, respectively; they occur when the derivative $d\sigma/dt$ vanishes. Consequently, applying the conditions $\sigma = \sigma_m$ and $\sigma = \sigma_o$ at $d\sigma/dt = 0$ to equation (13), we find E' and b to be given in terms of σ_m, σ_o, and P by the following expressions:

$$E' = \frac{(b - P\cos\sigma_m)^2}{\sin^2\sigma_m} + 2\sum_{n=o} \frac{M_n}{n+2}\sigma_m^{n+2} \tag{17a}$$

$$b = \frac{P}{\sin^2\sigma_m - \sin^2\sigma_o}\Bigg\{\sin^2\sigma_m\cos\sigma_o - \sin^2\sigma_o\cos\sigma_m$$

$$\pm \sin\sigma_m\sin\sigma_o\Bigg[(\cos\sigma_m - \cos\sigma_o)^2$$

$$+ \frac{2}{P^2}(\sin^2\sigma_m - \sin^2\sigma_o)\sum_{n=o}\frac{M_n}{n+2}\left(\sigma_m^{n+2} - \sigma_o^{n+2}\right)\Bigg]^{1/2}\Bigg\} \tag{17b}$$

Applying these conditions to equation (16), we get for a_o and $(E' - a_2)$

$$a_o = (b - P)^2 = \sum_{n=o}^{\infty}\frac{\bar{M}_n}{n+2}\left[\frac{\sigma_m^2 + \sigma_o^2}{\sigma_m^2 - \sigma_o^2}\left(\sigma_m^{n+4} - \sigma_o^{n+4}\right) - \left(\sigma_m^{n+4} + \sigma_o^{n+4}\right)\right] \tag{18a}$$

$$(E' - a_2) = \frac{2}{\sigma_m^2 - \sigma_o^2}\sum_{n=o}^{\infty}\frac{\bar{M}_n}{n+2}\left(\sigma_m^{n+4} - \sigma_o^{n+4}\right) \tag{18b}$$

Equations (17a) and (17b) are explicit relations for E' and b, and equations (18a) and (18b) are implicit relations. Substituting equations (18a) and (18b) into equation (16) and rearranging the terms, we can find the following symmetric form:

$$\frac{1}{8}\left(\frac{d\sigma^2}{dt}\right)^2 = \left(\sigma^2 - \frac{\sigma_m^2 + \sigma_o^2}{2}\right)\sum_{n=o}^{\infty}\frac{\bar{M}_n}{n+2}\,\frac{\sigma_m^{n+4} - \sigma_o^{n+4}}{\sigma_m^2 - \sigma_o^2}$$

$$+ \sum_{n=o}^{\infty}\frac{\bar{M}_n}{n+2}\left(\frac{\sigma_m^{n+4} + \sigma_o^{n+4}}{2} - \sigma^{n+4}\right) \tag{19}$$

The right-hand side of equation (19) is an infinite series in terms of powers of the resultant angle of attack σ. Each term of the series involves the difference between a power of the resultant angle and the average of the same powers of the maximum and minimum resultant angles.

If we make the transformation $\sigma^2 = \epsilon$, then equation (19) can be written

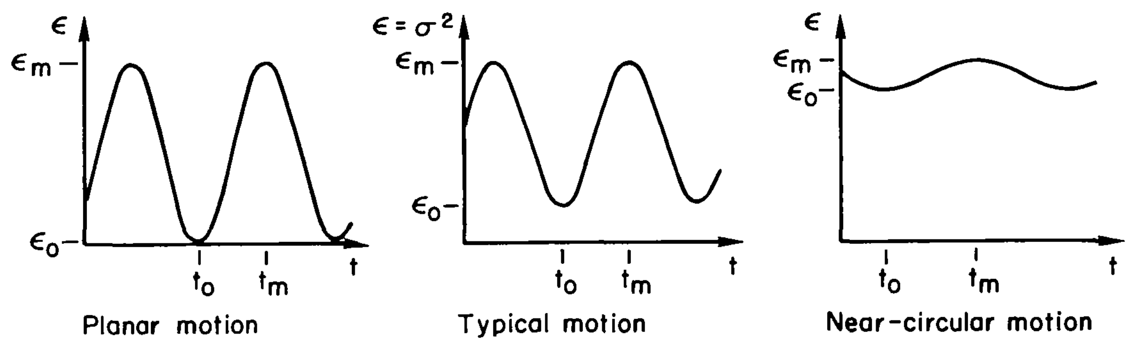
$$\frac{1}{8}\left(\frac{d\epsilon}{dt}\right)^2 = (\epsilon - \bar{\epsilon})\sum_{n=0}^{\infty}\frac{\bar{M}_n}{n+2}\,\frac{\sigma_m^{n+4} - \sigma_o^{n+4}}{\sigma_m^2 - \sigma_o^2} + \sum_{n=0}^{\infty}\frac{\bar{M}_n}{n+2}\left(\frac{\sigma_m^{n+4} + \sigma_o^{n+4}}{2} - \epsilon^{\frac{n+4}{2}}\right)$$

$$\equiv G_o(\epsilon) \qquad (20)$$

where $\bar{\epsilon} = (\sigma_m^2 + \sigma_o^2)/2$. One quarter of the period of oscillation may be found by integrating equation (20) from ϵ_o to ϵ_m, and thus the frequency ω may be given by the following quadrature:

$$t_m - t_o = \frac{\pi}{2\omega} = \frac{1}{\sqrt{8}}\int_{\epsilon_o}^{\epsilon_m}\frac{d\epsilon}{\sqrt{G_o(\epsilon)}} \qquad (21)$$

Various possible motions are shown in sketch (b).



Sketch (b)

The integral in equation (21) may be evaluated in closed form for certain cases by terminating the series given by $G_o(\epsilon)$ in equation (20) after a certain number of terms. The simplest case occurs when only the first term, corresponding to $n = 0$, is retained and the remaining terms neglected. This case corresponds to a linear restoring moment, $M = -M_o\sigma$; neglecting the higher order spinning terms, the frequency is found to be independent of σ_m and σ_o. The particular moment described by the cubic $M = -M_o\sigma - M_2\sigma^3$, where only terms given by $n = 0$ and $n = 2$ in the series were retained, was treated in reference 3. However, the spinning term a_6 in $\overline{M}_2$ was not taken into account. For this cubic moment the frequency may be evaluated in terms of a complete elliptic integral of the first kind. In principle, the frequency for the case where only the terms corresponding to $n = 0$, $n = 2$, and $n = 4$ are retained could also be evaluated in terms of elliptic integrals, but the solution would be very complicated. For an arbitrary restoring moment, an approximate method of evaluation must be developed.

An Approximate Expression for the Frequency

Since σ_m and σ_o are roots of the equation $G_o(\sigma^2) = 0$, we may rewrite equation (20) in the following form

$$G_o(\epsilon) = (\epsilon_m - \epsilon)(\epsilon - \epsilon_o)G_1(\epsilon) \tag{22}$$

where $G_1(\epsilon)$ is a well-behaved positive function in the interval $\epsilon_o \leq \epsilon \leq \epsilon_m$.

Equation (21) for the frequency may now be written

$$\frac{\pi}{\omega} = \frac{1}{\sqrt{2}} \int_{\epsilon_o}^{\epsilon_m} \frac{d\epsilon}{\sqrt{(\epsilon_m - \epsilon)(\epsilon - \epsilon_o)G_1(\epsilon)}} \tag{23}$$

Let us make a transformation to the new variable $\varphi = \epsilon - \overline{\epsilon}$, where $\overline{\epsilon} = (\epsilon_m + \epsilon_o)/2$. Expression (23) becomes

$$\frac{\pi}{\omega} = \frac{1}{\sqrt{2}} \int_{-\epsilon_1}^{\epsilon_1} \frac{d\varphi}{\sqrt{(\epsilon_1{}^2 - \varphi^2)G_1(\overline{\epsilon} + \varphi)}} \tag{24}$$

where $\epsilon_1 = (\epsilon_m - \epsilon_o)/2$.

The function $G_1(\epsilon)$ is essentially the function $G_o(\epsilon)$ with the two adjacent roots ϵ_m and ϵ_o factored out. Consequently, $G_1(\epsilon)$ will not vanish in the interval $\epsilon_o \leq \epsilon \leq \epsilon_m$ unless there are other roots equal to either ϵ_o or ϵ_m. Excluding this case, $G_1(\epsilon)$ will be nonvanishing over the interval $\epsilon_o \leq \epsilon \leq \epsilon_m$, and we can expand the square root of the inverse of $G_1(\epsilon)$ in a Taylor series about $\epsilon = \overline{\epsilon}$. Thus we have

$$\frac{\sqrt{G_1(\overline{\epsilon})}}{\sqrt{G_1(\overline{\epsilon} + \varphi)}} = 1 + B_1\varphi + B_2\varphi^2 + \ldots \tag{25}$$

where $B_1, B_2, B_3 \ldots$ are the respective coefficients of the Taylor expansion of $[G_1(\epsilon)]^{-1/2}$ multiplied by $\sqrt{G_1(\bar{\epsilon})}$. The above series is convergent in the interval $-\epsilon_1 \leq \varphi \leq \epsilon_1$, or $\epsilon_o \leq \epsilon \leq \epsilon_m$. The first two terms are given by

$$B_1 = -\left[\frac{1}{2G_1}\frac{dG_1}{d\epsilon}\right]_{\epsilon=\bar{\epsilon}} \tag{26a}$$

$$B_2 = \frac{1}{[2G_1(\bar{\epsilon})]^2}\left[\frac{3}{2}\left(\frac{dG_1}{d\epsilon}\right)^2 - G_1\frac{d^2G_1}{d\epsilon^2}\right]_{\epsilon=\bar{\epsilon}} \tag{26b}$$

Expression (24) can now be written as

$$\frac{\pi}{\omega} = \frac{1}{\sqrt{2G_1(\bar{\epsilon})}}\int_{-\epsilon_1}^{\epsilon_1}\frac{(1 + B_1\varphi + B_2\varphi^2 + \ldots)}{\sqrt{\epsilon_1^2 - \varphi^2}}\,d\varphi \tag{27}$$

and the integral can be integrated term by term. The odd functions of φ will all vanish, and the result of the integration is

$$\frac{\pi}{\omega} = \frac{\pi\left[1 + \frac{\epsilon_1^2}{2}B_2 + \frac{3}{8}\epsilon_1^4 B_4 + \ldots + \frac{1\cdot3\ldots(n-1)}{2\cdot4\ldots(n)}\epsilon_1^n B_n + \ldots\right]}{\sqrt{2G_1(\bar{\epsilon})}} \tag{28}$$

Inverting relation (28) and squaring, we can write the formula for the frequency as

$$\omega^2 = \frac{2G_1(\bar{\epsilon})}{\left[1 + \frac{\epsilon_1^2}{2}B_2 + \frac{3}{8}\epsilon_1^4 B_4 + \ldots + \frac{1\cdot3\ldots(n-1)}{2\cdot4\ldots(n)}\epsilon_1^n B_n + \ldots\right]^2} \tag{29}$$

Formula (29) is an exact expression for the frequency of oscillation for an arbitrary restoring moment. The convergence of the series in formula (29) can be demonstrated by making use of Taylor's formula with integral remainder, together with the information that the Taylor series given by (25) converges. A detailed proof is not given here. In principle, the value of the frequency could be determined to any degree of accuracy by taking enough terms of the series in the denominator of formula (29). This is generally difficult since the Taylor expansion coefficients $B_2, B_4, \ldots$ are complicated functions. The infinite series in formula (29), however, converges very rapidly even for the worst cases, and a remarkable approximation can be obtained by retaining only the first term of the series. The first-order approximation is thus given by

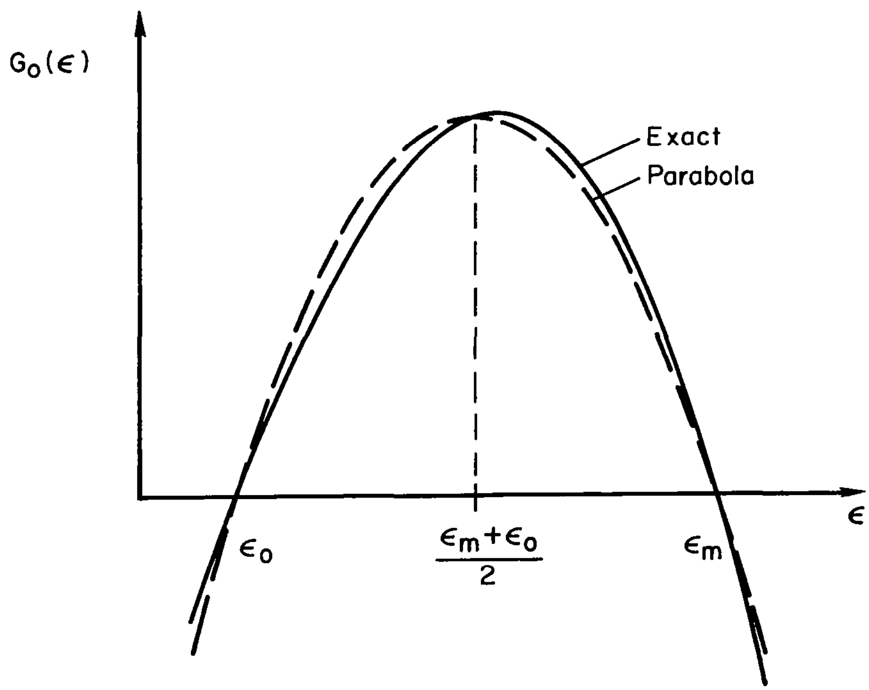
$$\omega^2 = 2G_1(\bar{\epsilon}) = \frac{8\sum_{n=o}^{\infty}\frac{\bar{M}_n}{n+2}\left[\frac{\sigma_m^{n+4} + \sigma_o^{n+4}}{2} - \left(\frac{\sigma_m^2 + \sigma_o^2}{2}\right)^{\frac{n+4}{2}}\right]}{(\sigma_m^2 - \sigma_o^2)^2} \tag{30}$$

Formula (30) is exact in the limit of circular motion. This can be seen from expression (29) by setting $\epsilon_1 = (\epsilon_m - \epsilon_o)/2 = 0$. It is also a good approximation even for planar motion. This can be deduced by examining the higher order approximations, which are discussed in the appendix. In addition, the frequency expression (30) can be compared with the Kryloff-Bogoliuboff technique, and this will be discussed in the next section.

The frequency expression (30) could have been obtained from a simpler point of view. Consider the curve of $G_o(\epsilon)$ as a function of ϵ in the interval $\epsilon_o \leq \epsilon \leq \epsilon_m$ shown in sketch (c). In order to evaluate approximately the integral given in equation (21), we could replace $G_o(\epsilon)$ by an equivalent parabola in the interval $\epsilon_o \leq \epsilon \leq \epsilon_m$, matching the parabola with the curve $G_o(\epsilon)$ at the three points ϵ_o, $(\epsilon_o + \epsilon_m)/2$, ϵ_m. The resulting parabola would be given by

$$(\epsilon_m - \epsilon)(\epsilon - \epsilon_o)G_1(\overline{\epsilon})$$

Substituting the parabola for $G_o(\epsilon)$ into equation (21) and integrating, we would obtain expression (30) for the frequency. The first derivation of formula (30), however, shows that it is the first approximation of a series expansion.



Sketch (c)

The frequency expression (30) contains the effects of nonlinear terms due both to the restoring moment nonlinearities, M_n, and to the nonlinearities due to spin. The nonlinearities due to spin are contained in the parameters a_n, specifically by the constants b and P. Both types of nonlinearities are represented by the parmeter $\overline{M}_n$ given by

$$\overline{M}_n = M_n + \frac{n + 2}{2} a_{n+4}$$

The term a_n may be determined in terms of the one spinning parameter P by using equation (17b) to evaluate b. An approximate formula for b in terms of P, however, may be obtained from equation (18a) for $a_o = (b - P)^2$, which is rewritten here.

$$a_o = (b - P)^2 = \sum_{n=o}^{\infty} \frac{\bar{M}_n}{n + 2} \left[\frac{\sigma_m^2 + \sigma_o^2}{\sigma_m^2 - \sigma_o^2}\left(\sigma_m^{n+4} - \sigma_o^{n+4}\right) - \left(\sigma_m^{n+4} + \sigma_o^{n+4}\right)\right] \quad (31)$$

For the case of zero minimum angle of attack, we have $\sigma_o = 0$ and, consequently, $a_o = 0$, or $b = P$. At the limit of circular motion, however, $\sigma_o = \sigma_m$, and using equation (31), we get for b

$$b = P \pm \left(\sum_{n=o}^{\infty} \bar{M}_n \sigma_m^{n+4}\right)^{1/2} \quad (32)$$

Thus, we can deduce, in general, that if

$$P \gg \left(\sum_{n=o}^{\infty} \bar{M}_n \sigma_m^{n+4}\right)^{1/2} \quad (33a)$$

we can make the general approximation $b = P$ and evaluate a_n in terms of P only. For this case, the first few values of $\bar{M}_n$ become

$$\bar{M}_o = M_o + \frac{P^2}{4} \quad (33b)$$

$$\bar{M}_1 = M_1 \quad (33c)$$

$$\bar{M}_2 = M_2 + \frac{P^2}{12} \quad (33d)$$

$$\bar{M}_3 = M_3 \quad (33e)$$

$$\bar{M}_4 = M_4 + \frac{17}{960} P^2 \quad (33f)$$

The spinning of the missile affects the frequency only through the even terms M_o, M_2, M_4, etc. These contributions to the terms M_2 and M_4 were not taken into account in references 1, 2, and 3. These effects, however, will generally be small.

Discussion of the Approximate Formula for the Frequency

The formulas given by (30) and (33) illustrate the influence of maximum and minimum amplitudes, the spinning rate, and the arbitrary restoring moment upon the frequency of the motion. The validity of equation (30) can be deduced by comparison with known exact results for certain limits of motion. In the limit of circular motion, of course, formula (30) is exact. This may be verified for the particular cubic moment $M = -M_0\sigma - M_2\sigma^3$, which was analyzed in reference 3 in terms of elliptic integrals.

Formula (30) may also be compared with the approximate results of reference 1 in which the Kryloff-Bogoliuboff method was used. In order to compare these results with the present method, we can write the first few terms of equation (30) as

$$\omega^2 = \overline{M}_0 + \frac{8}{3}\overline{M}_1 \frac{\dfrac{\sigma_m^{5} + \sigma_0^{5}}{2} - \left(\dfrac{\sigma_m^{2} + \sigma_0^{2}}{2}\right)^{5/2}}{(\sigma_m^2 - \sigma_0^2)^2}$$

$$+ \frac{3}{4}\overline{M}_2(\sigma_m^2 + \sigma_0^2) + \frac{8}{5}\overline{M}_3 \frac{\dfrac{\sigma_m^{7} + \sigma_0^{7}}{2} - \left(\dfrac{\sigma_m^{2} + \sigma_0^{2}}{2}\right)^{7/2}}{(\sigma_m^2 - \sigma_0^2)^2}$$

$$+ \frac{1}{12}\overline{M}_4(7\sigma_m^4 + 10\sigma_m^2\sigma_0^2 + 7\sigma_0^4)$$

$$+ \frac{8}{7}\overline{M}_5 \frac{\dfrac{\sigma_m^{9} + \sigma_0^{9}}{2} - \left(\dfrac{\sigma_m^{2} + \sigma_0^{2}}{2}\right)^{9/2}}{(\sigma_m^2 - \sigma_0^2)^2}$$

$$+ \frac{5}{32}\overline{M}_6(3\sigma_m^6 + 5\sigma_m^4\sigma_0^2 + 5\sigma_m^2\sigma_0^4 + 3\sigma_0^6) + \ldots \tag{34}$$

If the results of reference 1 are reduced to our nomenclature (for $P = 0$), we can obtain a similar expression for only the even terms:

$$\omega^2 = M_o + \frac{3}{4} M_2(\sigma_m^2 + \sigma_o^2) + \frac{1}{8} M_4(5\sigma_m^4 + 6\sigma_m^2\sigma_o^2 + 5\sigma_o^4)$$

$$+ \frac{5}{64} M_6(7\sigma_m^6 + 9\sigma_m^4\sigma_o^2 + 9\sigma_m^2\sigma_o^4 + 7\sigma_o^6) + \ldots \tag{35}$$

Comparing the results of formula (35) to formula (34), we can see that the terms corresponding to M_0 and M_2 are equal. The remaining even terms are different, but in the limit of circular motion ($\sigma_o = \sigma_m$), they reduce to the same value. The main differences will lie in the limit of vanishing minimum resultant angle, $\sigma_o = 0$ (e.g., planar motion). Hence, let us examine the frequency formula (30) in this limit in more detail. In the limit $\sigma_o = 0$, formula (30) assumes the simple form

$$\omega^2 = 4 \sum_{n=0}^{\infty} \frac{\bar{M}_n}{n + 2} \left[1 - \left(\frac{1}{2}\right)^{\frac{n+2}{2}}\right] \sigma_m^n \tag{36}$$

Several exact results can be obtained for planar motion. An analysis of planar motion is found in reference 4, and these results can be compared with equation (36). In addition, an approximate expression for the frequency can be easily obtained for planar motion from reference 6. This Kryloff-Bogoliuboff technique is used in reference 1.

A very simple comparison can be made with a one-term moment of arbitrary power of angle of attack

$$M = -M_n\alpha^{n+1}$$

The exact value for the frequency is given by reference 4, and is

$$\frac{\omega_n^2}{M_n\alpha_m^n} = 2\pi \frac{n + 2}{(n + 4)^2} \left[\frac{\Gamma\left(\frac{n + 3}{n + 2} + \frac{1}{2}\right)}{\Gamma\left(\frac{n + 3}{n + 2}\right)}\right]^2 \tag{37}$$

where $\Gamma(x)$ is the gamma function of argument x. The first-order Kryloff-Bogoliuboff approximation is given by (ref. 6)

$$\frac{\omega_n^2}{M_n\alpha_m^n} = \frac{2}{\pi} \int_0^{\pi} \sin^{(n+2)}\varphi \, d\varphi$$

$$= 2 \frac{1\cdot3\cdot5\ldots(n + 1)}{2\cdot4\cdot6\ldots(n + 2)}, \text{ n even}$$

$$= \frac{4}{\pi} \frac{2\cdot4\cdot6\ldots(n + 1)}{3\cdot5\cdot7\ldots(n + 2)}, \text{ n odd} \tag{38}$$

Using equation (36), we obtain for the present method

$$\frac{\omega_n^2}{M_n \alpha_m^n} = \frac{4}{n + 2} \left[1 - \left(\frac{1}{2}\right)^{\frac{n+2}{2}}\right] \tag{39}$$

Although equations (37), (38), and (39) are quite different in form, their values are very close to each other. The values of these functions for $n \leq 10$ are compared in the following table and figure 1.

	$\frac{\omega_n^2}{M_n \alpha_m^n}$		
n	Exact (eq. 37)	Present approx. (eq. 39)	Kryloff-Bogoliuboff (eq. 38)
0	1.0000	1.0000	1.0000
1	.8366	.8619	.8488
2	.7178	.7500	.7500
3	.6279	.6586	.6791
4	.5578	.5833	.6250
5	.5016	.5209	.5821
6	.4556	.4687	.5469
7	.4173	.4248	.5174
8	.3849	.3875	.4922
9	.3571	.3556	.4703
10	.3331	.3281	.4512

From this table and figure 1 we see that the values from the approximate formula (39) agree well with the exact values given by equation (37). In addition, the present approximation is superior to the often-used Kryloff-Bogoliuboff approximation for all values of n greater than 2. For $n = 2$, the Kryloff-Bogoliuboff and the present approximation give the same result. The Kryloff-Bogoliuboff approximation is superior only for $n = 1$, where it differs from the exact result by 1.5 percent, whereas the present approximation differs by 3.0 percent.

In the limit of planar motion, another comparison that can be made is for a linear-cubic-quintic restoring moment

$$M = -M_0\alpha - M_2\alpha^3 - M_4\alpha^5$$

For brevity, we will restrict the comparison to the case where the linear and quintic terms are stabilizing and the cubic term is destabilizing. The exact period of oscillation ($T_o = 2\pi/\omega$) will be compared with results from the present method and with similar results from the Kryloff-Bogoliuboff technique. The appropriate equations are

Present method:

$$\sqrt{M_o}\, T_o = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\frac{M_2}{M_o}\alpha_m^2 + \frac{7}{12}\frac{M_4}{M_o}\alpha_m^4}}$$

Kryloff-Bogoliuboff (ref. 6):

$$\sqrt{M_o}\, T_o = \frac{2\pi}{\sqrt{1 + \frac{3}{4}\frac{M_2}{M_o}\alpha_m^2 + \frac{5}{8}\frac{M_4}{M_o}\alpha_m^4}}$$

Exact (ref. 4):

$$\sqrt{M_o}\, T_o = \frac{2\pi}{\sqrt{1 + 3\frac{M_2}{M_o}\alpha_m^2\left(\frac{\alpha_E}{\alpha_m}\right)^2 + 5\frac{M_4}{M_o}\alpha_m^4\left(\frac{\alpha_E}{\alpha_m}\right)^4}}$$

where

$$\left(\frac{\alpha_E}{\alpha_m}\right)^2 = \frac{-3\frac{M_2}{M_o}\alpha_m^2 + \sqrt{\left(3\frac{M_2}{M_o}\alpha_m^2\right)^2 - 20\frac{M_4}{M_o}\alpha_m^4(1 - u)}}{10\frac{M_4}{M_o}\alpha_m^4}$$

$$u = \frac{\pi^2\frac{M_4}{M_o}\alpha_m^4}{12 g^2 [K(k)]^2}$$

$$g^2 = \frac{\frac{M_4}{M_o}\alpha_m^4}{\sqrt{\frac{3}{2}\left(1 + \frac{M_2}{M_o}\alpha_m^2 + \frac{M_4}{M_o}\alpha_m^4\right)\left(6 + 3\frac{M_2}{M_o}\alpha_m^2 + 2\frac{M_4}{M_o}\alpha_m^4\right)}}$$

$$k^2 = \frac{1}{2} - \frac{3}{4} \frac{4 + 3\frac{M_2}{M_0}\alpha_m^2 + 2\frac{M_4}{M_0}\alpha_m^4}{\sqrt{6\left(1 + \frac{M_2}{M_0}\alpha_m^2 + \frac{M_4}{M_0}\alpha_m^4\right)\left(6 + 3\frac{M_2}{M_0}\alpha_m^2 + 2\frac{M_4}{M_0}\alpha_m^4\right)}}$$

$K(k)$ = complete elliptic integral of the first kind

The comparisons are shown in the carpet plots in figure 2, where the dots represent the exact solution, and the solid lines, a given approximation. The Kryloff-Bogoliuboff approximation (fig. 2(a)) is seen to give a fairly good representation of the exact solution. Comparison of figure 2(b) with 2(a), however, shows the present method to be clearly superior. The accuracy obtained with the present method is much better than would have been casually anticipated. It is shown in the appendix that the present method is least accurate for the case of planar motion, so the favorable comparisons for the planar cases that have been shown justify a high confidence in the method in general.

Application to Free-Flight Data

Consider the determination of the restoring moment that governs a model in free flight when the frequency and amplitude have been measured for a large number of flights. Equation (30) provides a relation between the frequency and maximum and minimum amplitudes for an infinite number of possible moment combinations. We must attempt to find a moment that best represents the measured frequency of the motion over a given range of flight data.

Since there is an infinite number of combinations, this at first appears to be a hopeless task. Fortunately, however, a number of different assumed forms for the moment, when fit to the experimental data, give nearly identical results. Hence it is not necessary to find a unique moment that gives the best fit but only a member of a class of moments that gives a good fit.

These statements are justified as follows. An examination was made of the experimental data in reference 7, which showed large nonlinearities, and it was decided that a four-term moment in the resultant angle of attack would be necessary to adequately represent the data. Due to the absence of any low-amplitude data, the linear coefficient ($\overline{M}_0$) was assumed known, equal to the value given in reference 7. Then all possible four-term moments, each containing $\overline{M}_0$, and three members of the set ($\overline{M}_1$, $\overline{M}_2$, $\overline{M}_3$, $\overline{M}_4$, $\overline{M}_5$, $\overline{M}_6$) were fit to the experimental data using equation (34) and the method of least squares. The results are summarized as follows.

Powers of resultant angle of attack in assumed moment	Sum of the squares of the residuals
1-2-3-4	3.5×10^{-7}
1-2-3-5	4.9
1-2-3-6	6.6
1-2-3-7	8.5
1-2-4-5	6.6
1-2-4-6	8.4
1-2-4-7	10.4
1-2-5-6	10.2
1-2-5-7	11.6
1-2-6-7	12.4
1-3-4-5	2.4
1-3-4-6	2.0
1-3-4-7	1.9
1-3-5-6	1.8
1-3-5-7	2.0
1-3-6-7	3.2
1-4-5-6	8.4
1-4-5-7	11.3
1-4-6-7	17.3
1-5-6-7	32.0×10^{-7}

It is easy to see that most of the moments need not be considered because of the large error sum relative to the better fits. The important question is how to choose between these better fits that have about the same error. Figure 3 shows the envelope from the five moments that gave the smallest sum of the squares of the residuals. The moment curves are close enough together that no real choice between them need be made. Figure 3 also shows a substantial difference above 16° between these moments and the moment from reference 7. The greater number of nonlinearities that can be treated with the present method allows a more precise determination of the restoring moment than the linear plus cubic segmented approximation used in reference 7.

CONCLUSIONS

An analysis has been presented of the pitching and yawing motion of a spinning symmetric missile acted upon by a restoring moment represented by an arbitrary power series of the resultant angle of attack. The following conclusions were obtained:

1. The approximate solution obtained for the frequency of oscillation is least accurate for the case of planar motion, becomes increasingly more accurate as the motion becomes more and more circular, and is exact in the limit of circular motion.

2. For the case of planar motion, the approximate solution gives results that are extremely close to results of exact solutions.

3. The approximate solution gives results that are, for the most part, more accurate than results given by the Kryloff-Bogoliuboff technique.

Ames Research Center
National Aeronautics and Space Administration
Moffett Field, Calif., Nov. 27, 1963

APPENDIX

EFFECT OF THE HIGHER ORDER TERMS ON THE VALUE OF THE FREQUENCY FORMULA

Formula (29) for the frequency of oscillation is rewritten here

$$\omega^2 = \frac{2G_1(\bar{\epsilon})}{\left[1 + \frac{\epsilon_1^2}{2} B_2 + \frac{3}{8}\epsilon_1^4 B_4 + \frac{5}{16}\epsilon_1^6 B_6 + \dots\right]^2} \tag{A1}$$

The approximate formula (30)

$$\omega^2 = 2G_1(\bar{\epsilon}) \tag{A2}$$

can be considered the first-order term in a system of higher order approximations. The second-order approximation would be

$$\omega^2 = \frac{2G_1(\bar{\epsilon})}{\left[1 + \frac{\epsilon_1^2}{2} B_2\right]^2} \tag{A3}$$

and so on for the higher approximations.

We can consider the effect of the second approximation by considering the term B_2 in more detail.

$$B_2 = \frac{\left[\frac{3}{2}\left(\frac{dG_1}{d\epsilon}\right)^2 - G_1 \frac{d^2G_1}{d\epsilon^2}\right]_{\epsilon=\bar{\epsilon}}}{[2G_1(\bar{\epsilon})]^2} \tag{A4}$$

The function $G_1(\bar{\epsilon})$ and its derivatives in the expression for B_2 are

$$G_1(\bar{\epsilon}) = \frac{4\sum_{n=0}^{\infty} \frac{\bar{M}_n}{n+2}\left(\frac{\sigma_m^{n+4} + \sigma_o^{n+4}}{2} - \bar{\epsilon}^{\frac{n+4}{2}}\right)}{(\sigma_m^2 - \sigma_o^2)^2} \tag{A5a}$$

$$\left(\frac{dG_1}{d\epsilon}\right)_{\epsilon=\bar{\epsilon}} = \frac{4\sum_{n=0}^{\infty}\frac{\bar{M}_n}{n+2}\left(\frac{\sigma_m^{n+4}-\sigma_o^{n+4}}{\sigma_m^2-\sigma_o^2}-\frac{n+4}{2}\bar{\epsilon}^{\frac{n+2}{2}}\right)}{(\sigma_m^2-\sigma_o^2)^2} \tag{A5b}$$

$$\left(\frac{d^2G_1}{d\epsilon^2}\right)_{\epsilon=\bar{\epsilon}} = \frac{8G_1(\bar{\epsilon}) - \sum_{n=0}^{\infty}\bar{M}_n(n+4)\bar{\epsilon}^{\frac{n}{2}}}{(\sigma_m^2-\sigma_o^2)^2} \tag{A5c}$$

The factor $(\sigma_m^2 - \sigma_o^2)^2$ appears in the denominator of each of the terms (A5a, b, c), but in the combination of these terms given by the expression for B_2, this factor cancels, thus B_2 will have a finite value as $\sigma_o \to \sigma_m$. The factor ϵ_1 vanishes as $\sigma_o \to \sigma_m$, however, and thus (A3) reduces to (A2) in the limit of circular motion.

We can get the best physical significance for B_2 by considering the case of a one-term moment given by

$$M = -M_n\sigma^{n+1} \tag{A6}$$

Letting $\gamma = \sigma_o/\sigma_m$, we can then express $G_1(\bar{\epsilon})$ and its derivatives as

$$G_1(\bar{\epsilon}) = \frac{\frac{4\bar{M}_n\sigma_m^n}{n+2}\left[\frac{1+\gamma^{n+4}}{2} - \left(\frac{1+\gamma^2}{2}\right)^{\frac{n+4}{2}}\right]}{(1-\gamma^2)^2} \tag{A7a}$$

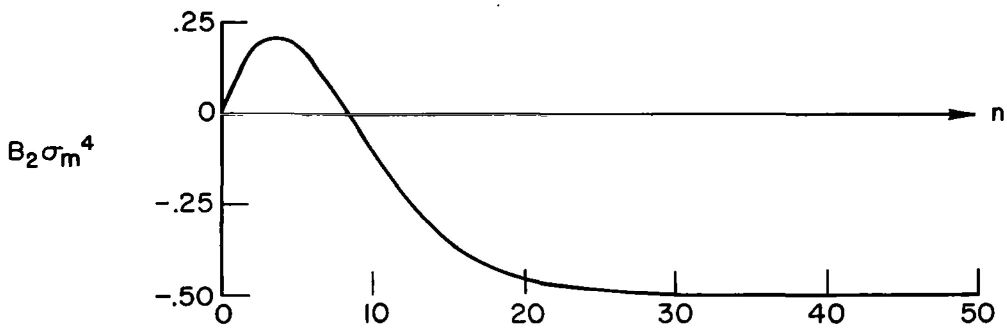
$$\left(\frac{dG_1}{d\epsilon}\right)_{\epsilon=\bar{\epsilon}} = \frac{\frac{4\bar{M}_n\sigma_m^{n-2}}{n+2}\left[\frac{1-\gamma^{n+4}}{1-\gamma^2} - \left(\frac{n+4}{2}\right)\left(\frac{1+\gamma^2}{2}\right)^{\frac{n+2}{2}}\right]}{(1-\gamma^2)^2} \tag{A7b}$$

$$\left(\frac{d^2G_1}{d\epsilon^2}\right)_{\epsilon=\bar{\epsilon}} = \frac{\frac{8}{\sigma_m^4}G_1(\bar{\epsilon}) - \bar{M}_n\sigma_m^{n-4}\left[(n+4)\left(\frac{1+\gamma^2}{2}\right)^{\frac{n}{2}}\right]}{(1-\gamma^2)^2} \tag{A7c}$$

Now consider the case of planar motion ($\gamma = 0$). We then get a fairly simple expression for B_2 by substituting equations (A7) into (A4).

$$B_2 = \frac{1}{8\sigma_m^4}\left[\frac{n^2\left(1 + 2^{\frac{n+4}{2}}\right) + 12n + 16(1 - 2^n)}{4(2)^n - 4(2)^{\frac{n}{2}} + 1}\right] \tag{A8}$$

The investigation of equation (A8) leads to several interesting results. The quantity B_2 as a function of n is shown in sketch (d), and B_2 is largest in absolute value when $n \to \infty$, this value being $B_2 = -1/2\sigma_m^4$.



Sketch (d)

Since for planar motion $\epsilon_1^2 = \sigma_m^4/4$, we are left with

$$\lim_{n \to \infty} \frac{\epsilon_1^2}{2} B_2 = -\frac{1}{16}$$

The greatest contribution that $(\epsilon_1^2/2)B_2$ can make in equation (A3) is thus about $-1/16$. This most extreme case is a relatively small contribution.

We now return to the cases of nonplanar motion ($\gamma \neq 0$). Again equations (A7) are substituted into (A4), but the resulting expression for B_2 is too cumbersome to write out. What is found, however, is that in <u>all</u> cases (except $\gamma = 1$), B_2 is largest in absolute value when $n \to \infty$, and in all cases

$$\lim_{n \to \infty} \frac{\epsilon_1^2}{2} B_2 = -\frac{1}{16} \qquad (\gamma \neq 1)$$

Hence, the <u>extreme</u> contribution of the term involving B_2 does not depend on the type of motion.

We can now summarize results that were obtained for higher order terms. It was found that regardless of how many of the B_n terms were retained, the worst case was when $n \to \infty$.

$$\lim_{n \to \infty} \frac{3}{8} \epsilon_1^4 B_4 = - \frac{15}{1024}$$

$$\lim_{n \to \infty} \frac{5}{16} \epsilon_1^6 B_6 = - \frac{105}{16384}$$

Define

$$u_1 = 1 + \frac{\epsilon_1^2}{2} B_2$$

$$u_2 = 1 + \frac{\epsilon_1^2}{2} B_2 + \frac{3}{8} \epsilon_1^4 B_4$$

$$u_3 = 1 + \frac{\epsilon_1^2}{2} B_2 + \frac{3}{8} \epsilon_1^4 B_4 + \frac{5}{16} \epsilon_1^6 B_6$$

Then

$$\lim_{n \to \infty} u_1 = 0.9375$$

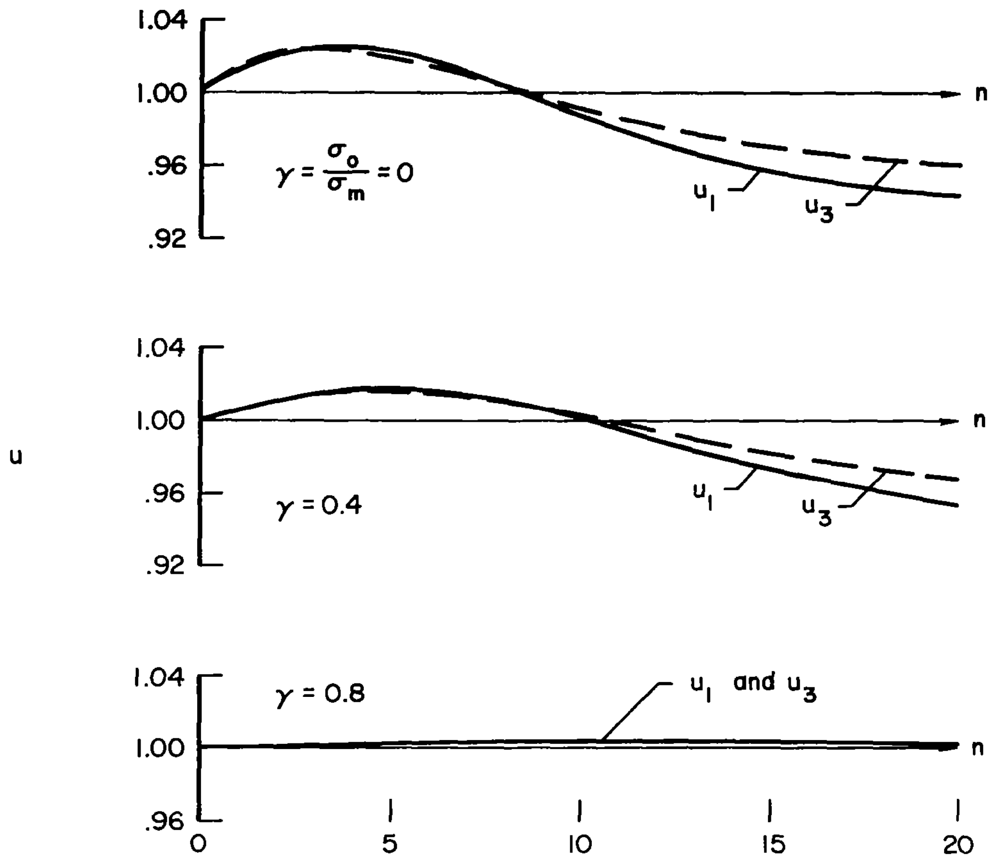
$$\lim_{n \to \infty} u_2 = 0.9229$$

$$\lim_{n \to \infty} u_3 = 0.9164$$

$$\vdots$$

$$\lim_{n \to \infty} u_n = 0.9003$$

So far we have considered the extreme contributions of the terms in equation (A1) involving the B_n. Next, we must consider the realistic contributions, where we restrict the exponent in equation (A6) to a reasonable value, like $n \leq 20$. When this is done, we find that the value of $\gamma = \sigma_o/\sigma_m$ significantly influences the size of the B_n contribution, and, moreover, that the contributions are much smaller than in the extreme case. Sketch (e) shows results for several values of γ. The first and third corrections, u_1 and u_3, are shown as functions of n, u_2 being omitted for clarity.



Sketch (e)

If the u's in the sketch were identically unity, this would mean that the approximate formula (A2) was exact. It is noted that the deviation from unity is very small. It is also noted that the deviation from unity for the three-term correction, u_3, is generally smaller than that for the single term correction, u_1. The first correction alone overestimates the correction to the approximate formula (A2). Finally, it can be seen from this sketch that the approximate formula is poorest for planar motion and becomes increasingly better as σ_o/σ_m increases toward 1. Although these results are for a single-term moment, they would be expected to be qualitatively true for more complex moments.

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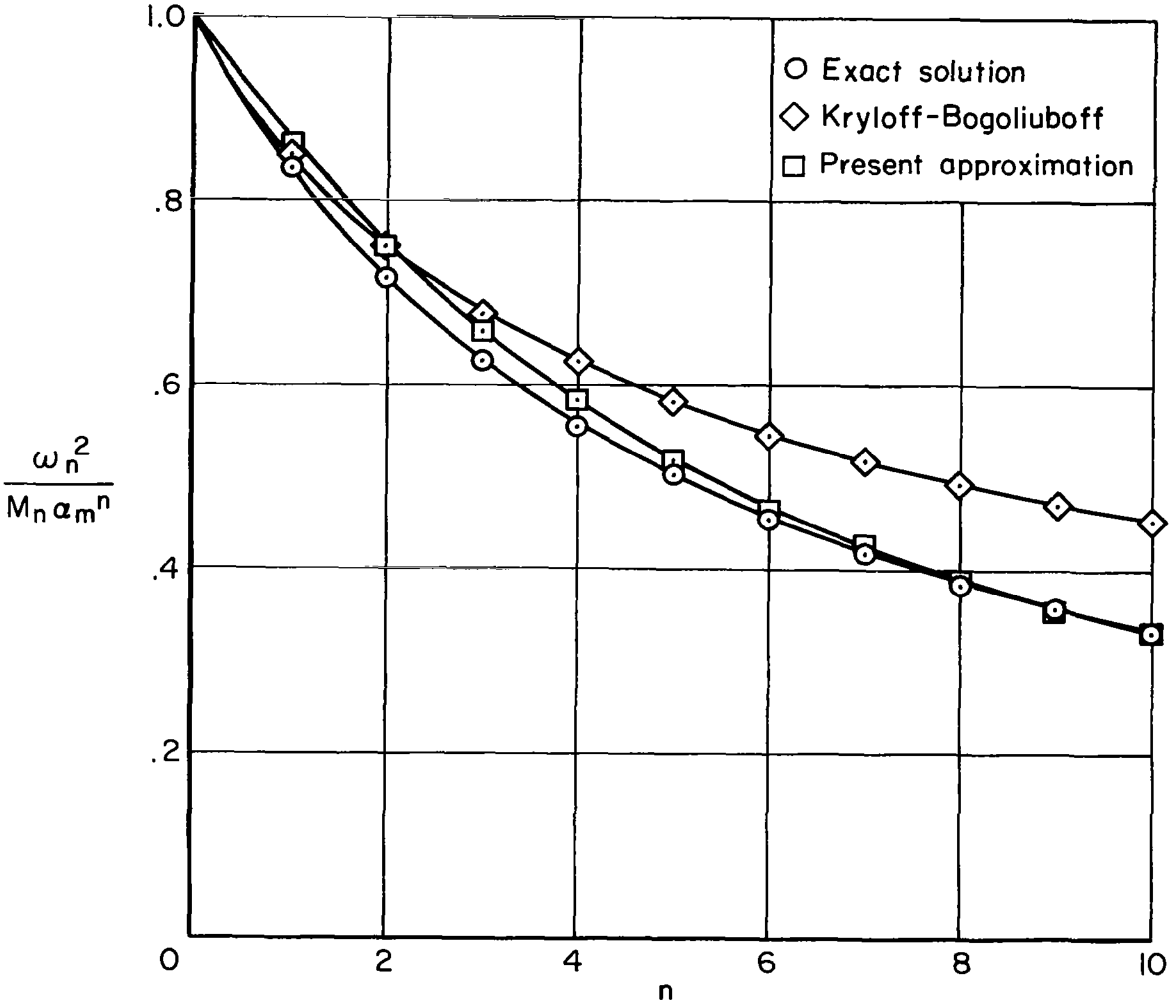
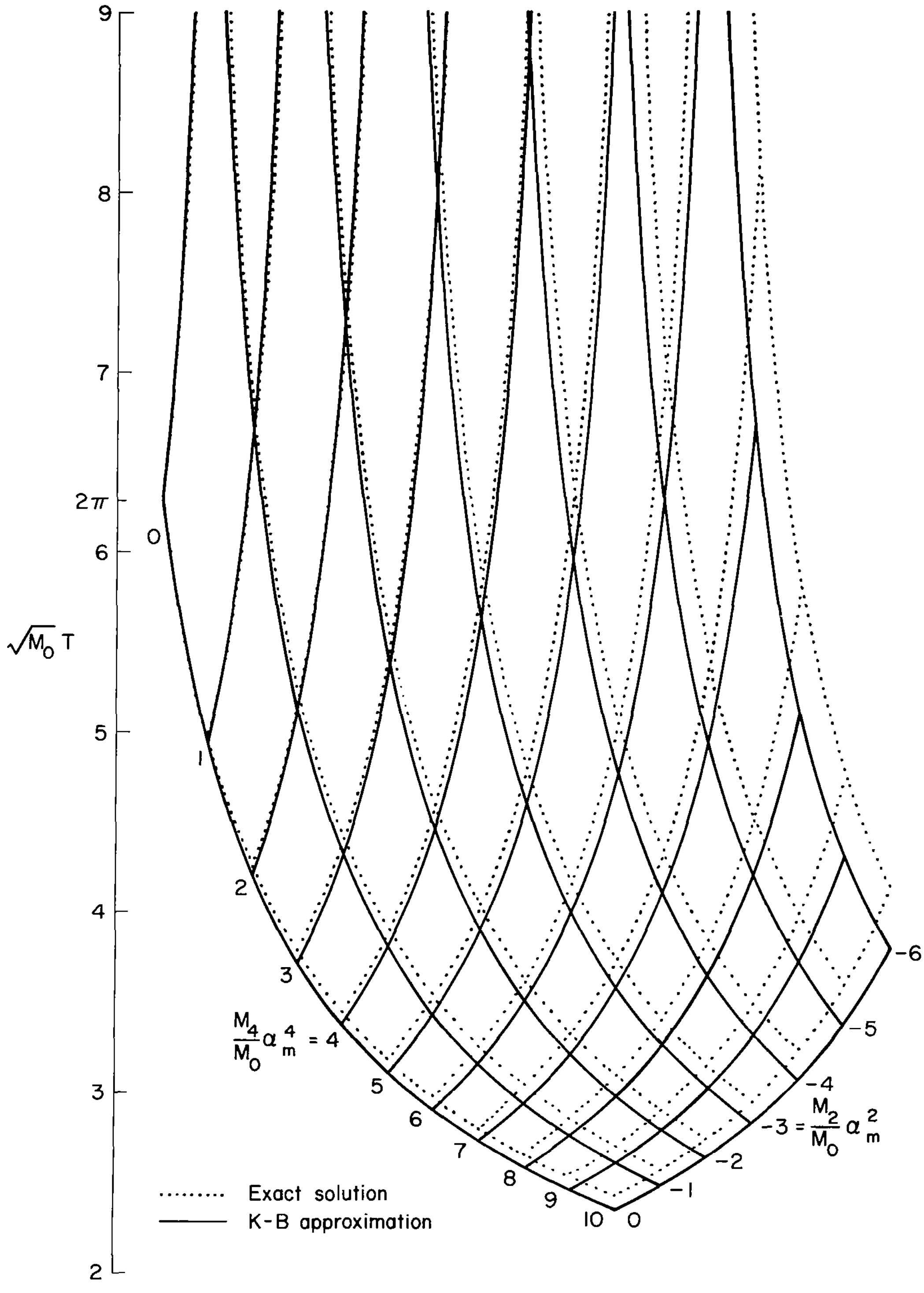
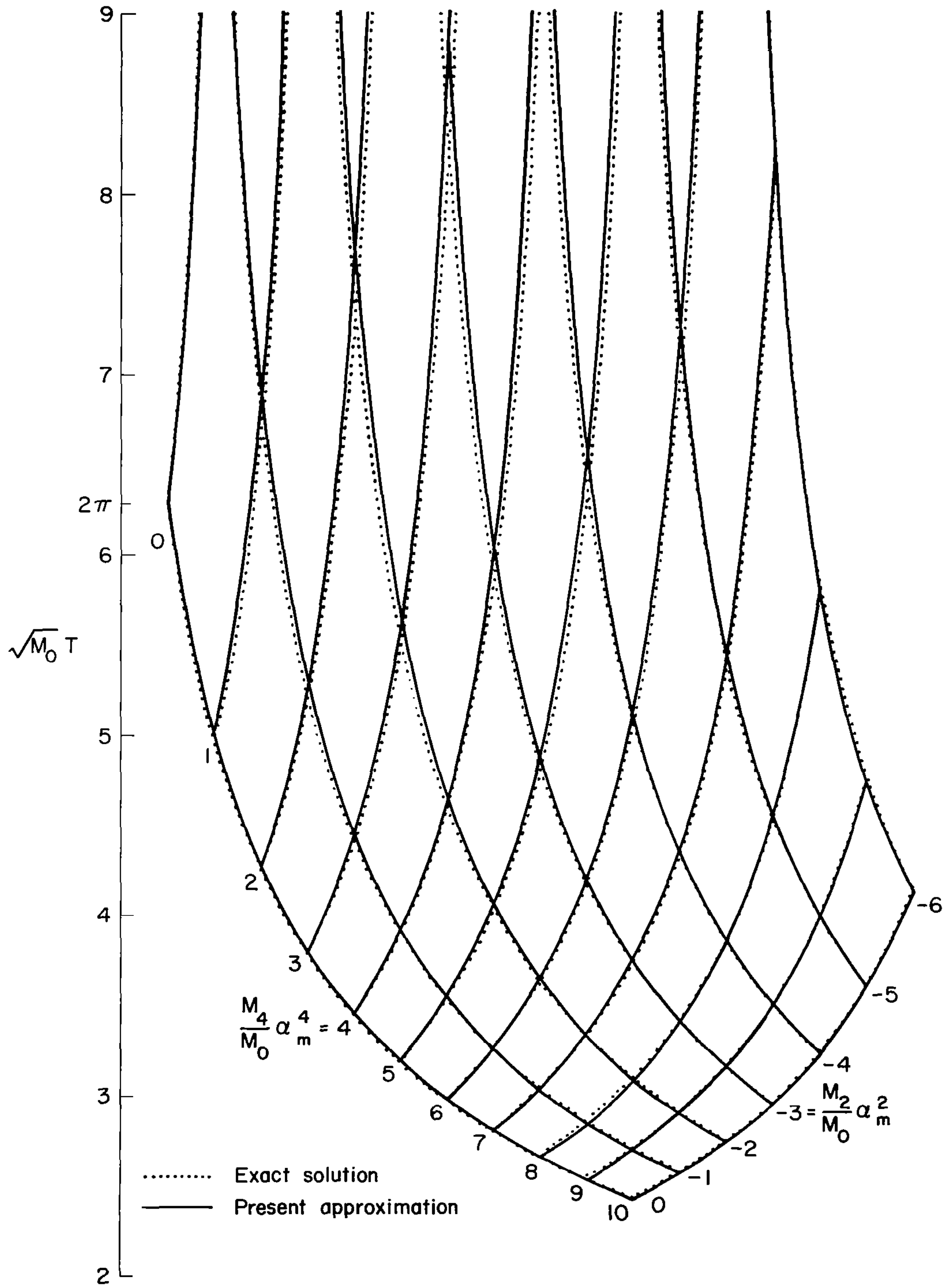


Figure 1.- Comparison of approximate solutions with exact solution; $M = -M_n\alpha^{n+1}$.



(a) K-B approximation versus exact solution.

Figure 2.- Carpet plot showing comparison of approximate solutions with exact solution; $M = -M_0\alpha - M_2\alpha^3 - M_4\alpha^5$, $M_0 > 0$, $M_2 < 0$, $M_4 > 0$.



(b) Present approximation versus exact solution.

Figure 2.- Concluded.

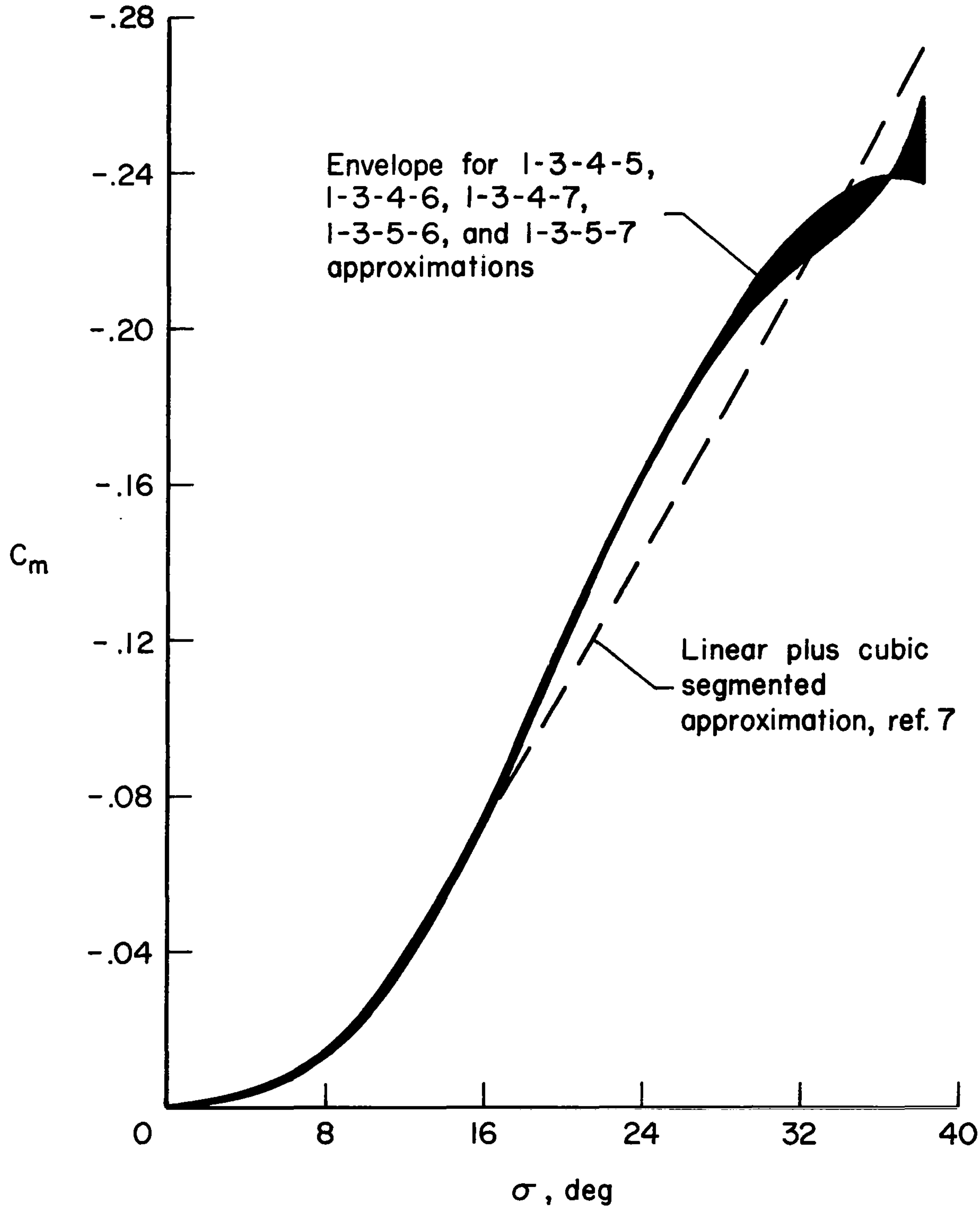


Figure 3.- Restoring moment coefficient corresponding to various assumed moment representations.

NASA-Langley, 1964 A-797